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An appreciation of advances in natural convection along an isothermal vertical surface

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Abstract—As a tribute to Sir Owen Saunders, this article traces developments in our knowledge of natural convection along an isothermal vertical surface from their beginnings over a century ago. Saunders' contributions are seen to have been crucial, first in developing wholly-theoretical heat transfer predictions for laminar flow which were applicable to other than Pr near unity. These predictions agreed well with his own measurements for mercury, the first reported for this fluid, and his earlier comprehensive measurements for air over a wide Ra range. Thus Saunders was also the first to report precisely the onset of transition, the now well-established power law governing turbulent heat transfer and subsequently the reduced effect of viscosity in pronounced turbulence. Later workers stand much in his debt for the foundations thereby laid.

THE SIMPLEST example of natural convection is that of fluid motion adjacent to a plane vertical surface at uniform temperature. More than a century ago, in 1879, Oberbeck [1] made the first attempt to calculate the heat transfer for laminar flow by power series solution in β of the general conservation equations. In a pioneer paper two years later, and more than 40 years before the first measurements were reported by Griffiths and Davis [2] for atmospheric air, Lorenz [3] reduced the number of Oberbeck's equations to two by certain rather sweeping assumptions and thereby obtained a solution.

Lorenz erroneously assumed that the flow in convection layers was essentially parallel to the surface, and derived the thickness of a layer from a force balance which included the effects of buoyancy and fluid shear. The surface heat transfer coefficient was obtained by treating the convection layer as a thin slab across which heat was conducted. In modern terminology Lorenz predicted that the average surface heat transfer coefficient could be found from

$$Nu/Ra^{1/4} = c \tag{1}$$

where $c = 0.548$. This value, which is independent of Prandtl number, is nevertheless valid for Pr near unity, as shown in Fig. 1. The form of equation (1) is that predicted from similarity considerations.

Nearly 30 years after Prandtl developed his boundary-layer concept the resultant approximations were adapted by Pohlhausen [4] to analyse laminar natural convection. His transformation of the governing partial differential equations of continuity, momentum and energy (which were in fact the Oberbeck equations in less primitive form) into equations with the single independent variable $y/x^{1/4}$, involved the specification of three boundary conditions at the surface and two more at infinity. Solution by power series resulted in such slow convergence that Pohlhausen replaced the boundary conditions at infinity by two more at the surface. These were the normal velocity and temperature gradients measured by Schmidt and Beckmann, but for air only. Pohlhausen thus obtained satisfactory convergence of a series solution which also satisfied conditions at infinity. This yielded for air and other fluids of $Pr = 0.72$, the correct value for c of 0.517 based on the average heat transfer, as indicated in Fig. 1.

It clearly remained to broaden theoretical investigation to other Pr by procedures which yielded predictions without recourse to measurements. Saunders [5] reduced the order of Pohlhausen's transformed equations by making the dimensionless temperature the independent variable; the dependent variable was the temperature gradient with respect to $y/x^{1/4}$. Saunders then needed to solve only one fourth-order total differential equation. This procedure also reduced the number of boundary conditions to two at the surface and one at infinity, all for temperature. Approximate solutions were obtained for various Pr by a polynomial series of increasing degree corresponding to satisfaction of boundary conditions and the differential equation at an increasing number of values in the temperature range.

The heavy numerical work involved limited investigation to fifth-degree polynomials, a limitation which would have been removed with modern computers. Saunders' predictions of $c = Nu/Ra^{1/4}$ are displayed as a function of Pr in Fig. 1. For $Pr = 0.03$ his prediction of $c = 0.33$ agreed well with his measurements using mercury of 0.31 and 0.35 for $10^{5.7} < Ra < 10^{7.0}$. His earlier measurements for air [6], almost certainly the most precise and comprehensive set ever made for this fluid, yielded $c = 0.516$ for Ra up to $10^{9.3}$, which was rather closer to Pohlhausen's prediction than his own of 0.50. Lorenz's [7] measurements for oil of unspecified Pr gave $c = 0.56$ and thus supported Saunders' prediction of 0.54 for $Pr > 10$.

NOMENCLATURE

c	numerical coefficient	ΔT	driving temperature difference between surface and undisturbed fluid
C_p	fluid specific heat at constant pressure	x	distance along surface
g	external acceleration	y	distance normal to surface.
k	fluid thermal conductivity		
l	height of vertical surface	Greek symbols	
Nu	Nusselt number, $Ql/k\Delta T$	β	coefficient of cubical expansion of fluid
Pr	fluid Prandtl number, ν/κ	κ	fluid thermal diffusivity
Q	mean heat transfer rate per unit area	ν	fluid kinematic viscosity.
Ra	Rayleigh number, $\beta g l^3 \Delta T/\nu\kappa$		

Almost contemporaneously with Saunders [5], Squire [8] adopted an alternative analytical approach whereby the momentum and energy partial differential equations were integrated over the boundary layer. His approximate treatment then involved choosing velocity and temperature profiles to satisfy three boundary conditions at the surface and four more at the edge of the boundary layer. Thus the velocity profile was modelled by a cubic polynomial and the temperature profile by a quadratic. In contrast to Saunders' implicit procedure, that of Squire resulted in the following explicit relation between c and Pr

$$c = Nu/Ra^{1/4} = \frac{0.677}{(1+20/21Pr)^{1/4}}. \quad (2)$$

Figure 1 shows that except for liquid metals, equation (2) predicts significantly larger c than those of Saunders, giving 0.549 for air against Pohlhausen's value of 0.517. Though originally unpublished, Squire's treatment has since been given by Eckert and Gross [9] and others.

Further progress was interrupted by the Second World War and nearly a decade was to pass before Schuh [10] presented further wholly theoretical solutions of the Pohlhausen equations by the approximate methods followed by Saunders. With the advent of the computer a few years later, Ostrach [11] finally obtained exact solutions of the same laminar equations for eight values of Pr from 0.01 to 1000, as shown in Fig. 1, with similar calculations by Sugawara and Michiyoshi [12], Sparrow *et al.* [13] and Gebhart [14]. The asymptotic behaviour of the equations for very large and very small Pr was determined by Le Fevre [15] and confirmed by the matched asymptotic calculations of Kuiken [16, 17]. For $Pr > 100$, the inertial terms can justifiably be omitted from the momentum equations and for $Pr \to 0$, viscosity should not appear in the general solution.

Ostrach's values of c are only marginally greater than those of Saunders for liquid metals in Fig. 1 but they become increasingly higher as Pr increases above unity and ultimately approach those of Squire. His integral procedure was extended to higher-degree polynomial and exponential expressions for the velocity and temperature profiles by the later work of Sugawara and Michiyoshi [18], Merk and Prins [19] and Fujii [20]. Le Fevre [15] proposed that Ostrach's exact predictions be correlated by

$$c = \left[\frac{Pr}{2.43+4.88Pr^{1/2}+4.95Pr}\right]^{1/4}. \quad (3)$$

More recently Churchill and Chu [21] have

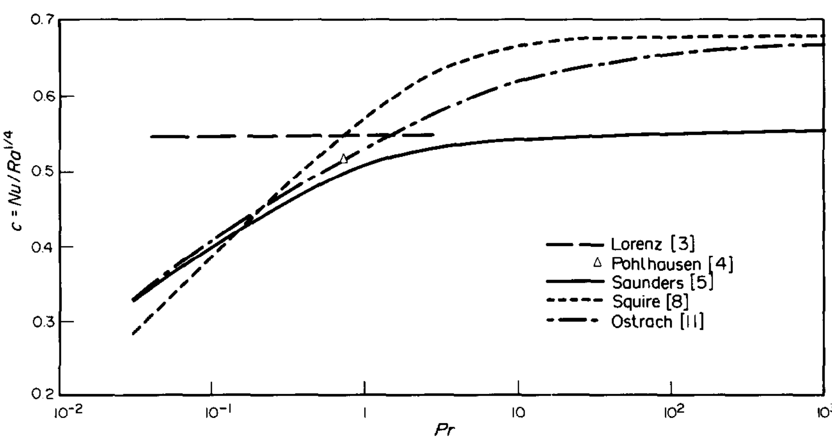

FIG. 1. Heat transfer predictions for natural convection along an isothermal vertical surface.

recommended that the effect of Ra be taken into account by the expression

$$c = \left[\frac{0.68}{Ra^{1/4}} + \frac{0.67}{1+(0.492/Pr)^{9/16}}\right]^{4/9} \quad (4)$$

for $Ra < 10^9$. Near this value, where the influence of Ra on c is least, equation (4) can be regarded as a refinement of equation (2); then both equations (3) and (4) agree closely with Ostrach's solutions except in the case of liquid metals, where compared also with Saunders' [5] measurements for mercury, c is underpredicted by about 9%.

The instability marking transition from laminar to turbulent natural convection was first observed for a vertical surface in atmospheric air by Griffiths and Davis [2] at Ra between $10^{8.08}$ and $10^{9.0}$. Saunders [6] reported the onset of transition in air at pressures up to 1000 bar at the somewhat higher value of $10^{9.23}$, which he attributed to steadier conditions in his enclosure. He found virtually the same value of $10^{9.3}$ for water, both from heat transfer measurements and from an optical refraction method [22] in which flow unsteadiness was detected by the variation in deflection of a light beam due to temperature gradients in the fluid.

It is a testimony to the accuracy of Saunders' work that the later observations of the onset of transition compiled by Mahajan and Gebhart [23] of Warner and Arpaci [24], Colak-Antic [25], Cheesewright [26], Regnier and Kaplan [27], Eckert and Soehngen [28], Hugot *et al.* [29] and Szewczyk [30], all but the last for air and some by interferometric studies, average out at $Ra = 10^{9.28}$. In one of the most recent theoretical contributions to the understanding of transition, these studies were used by Bejan and Cunnington [31] to support their argument that instability occurs when the fluctuating time period of the unstable (inviscid) wall jet is of the same order as the viscous diffusion time normal to the jet; their transition criterion $Ra^{1/4}/58Pr^{1/2}$ is then of order unity. Saunders' [6] measurements yield values of the criterion of 4.13 for air and 1.38 for water.

Saunders' [5, 6] observations showed that when the transition to turbulent flow along an isothermal vertical surface was complete, Nu increased more rapidly with Ra than in laminar flow, such that

$$Nu/Ra^{1/3} = c. \quad (5)$$

For air, with $Pr = 0.72$, Saunders found that over the range $10^{10} \leqslant Ra \leqslant 10^{11.6}$, $c = 0.10$, a result in conformity with the later implicit predictions of Bayley [32]. His treatment involved an eddy diffusivity, where Prandtl's mixing-length theory was used in conjunction with a turbulent boundary-region velocity distribution, and with postulated boundary conditions on the eddy diffusivity. For water, with $Pr = 7.0$, Saunders found c to have the higher value of 0.17 for $10^{10} \leqslant Ra \leqslant 10^{10.7}$.

Subsequent experience with turbulent boundary layers was that at higher Ra, Nu increased even more rapidly than as given by equation (5). This was predicted by Eckert and Jackson [33] using an integral method of analysis, with assumed forms of turbulent temperature and velocity distributions. The absolute velocity and surface shear stress were calculated from known forced-flow turbulent boundary-layer characteristics. This analysis gave

$$Nu/Ra^{2/5} = \frac{0.0246Pr^{1/15}}{(1+0.494Pr^{2/3})^{2/5}} \quad (6)$$

which correlated Saunders' (and certain other) measurements for air in the same Ra range almost as well as equation (5). However, agreement was less satisfactory for Pr markedly different from unity and for $Pr > 5$, soon led to errors in excess of 10%.

It is hoped that the foregoing sets in context, and thus provides some appreciation of, Saunders' contribution to our knowledge of natural convection. His work in this field alone not only laid a sound foundation for further advances; it also gave valuable insights into related areas of natural convection, some with more complex flow regimes. One such was the open thermosyphon, initially contemplated for cooling turbine blades, where the opposing fluid streams have to create an internal boundary. Lighthill's [34] treatment for the limiting case of very high Ra, when the boundary layer does not fill the thermosyphon cavity, was to extrapolate Saunders' [5, 6] empirical formulae for a vertical surface. For $Pr \gg 1$, the author's [35] measurements identified laminar and turbulent boundary-layer regimes analogous to those found by Saunders, Nu being related to Ra by power laws of similar form. In pronounced turbulence, when the opposing steams were fully mixed, correlation of heat transfer measurements for different fluids could only be achieved by using a modified parameter $Ra\ Pr^{0.55}$ as proposed by Fishenden and Saunders [36] in order to take account of the reduced influence of viscosity.

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APPRECIATION DES CONNAISSANCES SUR LA CONVECTION NATURELLE LE LONG D'UNE SURFACE VERTICALE ISOTHERME

Résumé—En hommage à Sir Owen Saunders, cet article retrace les développements de la connaissance sur la convection naturelle le long d'une surface verticale isotherme, depuis le début il y a cent ans. Les contributions de Saunders sont considérées comme cruciales, tout d'abord en développant la théorie complète du transfert thermique en écoulement laminaire, applicable à *Pr* autre que proche de l'unité. Ces calculs s'accordent bien avec ses propres mesures antérieures sur l'air dans un large domaine de *Ra*. Saunders a été le premier à traiter avec précision l'apparition de la transition, la loi puissance maintenant bien établie qui gouverne le transfert turbulent et aussi l'effet réducteur de la viscosité dans la forte turbulence. Les chercheurs ultérieurs lui doivent beaucoup pour les fondements qui sont bien posés grâce à lui.

EINE WÜRDIGUNG DER FORTSCHRITTE BEI DER ERFORSCHUNG DER NATÜRLICHEN KONVEKTION AN EINER SENKRECHTEN ISOTHERMEN FLÄCHE

Zusammenfassung—In Anerkennung von Sir Owen Saunders zeichnet dieser Artikel die Entwicklung unseres Wissens über die natürliche Konvektion on einer isothermen senkrechten Fläche von den Anfängen vor über einem Jahrhundert nach. Die Beiträge Saunder's waren von herausragender Bedeutung, vor allem bei der Entwickling rein theoretischer Modelle für den Wärmeübergang in laminarer Strömung, die für von eins abweichende Prandtl-Zahlen anwendbar waren. Diese Rechenergebnisse stimmten gut mit seinen eigenen Messungen an Quecksilber überein, die ersten, die mit diesem Stoff durchgeführt wurden, und mit seinen früheren ausführlichen Messungen mit Luft, die sich über einen großen Bereich der Rayleigh-Zahl erstreckten. So hat Saunders auch als erster exakt das Einsetzen des Überganges, das inzwischen wohlbekannte Potenzgesetz für den Wärmeübergang im turbulenten Bereich und in der Folge den verminderten Einfluß der Zähigkeit bei voll ausgebildeter Turbulenz beschrieben. Alle, die später daran gearbeitet haben, stehen für die von ihm geschaffenen Grundlagen in seiner Schuld.

О ДОСТИЖЕНИЯХ В ИССЛЕДОВАНИИ ПРОБЛЕМЫ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ У ИЗОТЕРМИЧЕСКОЙ ВЕРТИКАЛЬНОЙ ПОВЕРХНОСТИ

Аннотация—В знак признания заслуг сэра Оуэна Саундерса в предлагаемой работе прослеживаются этапы исследований проблемы естественной конвекции у изотермической вертикальной поверхности, начало которым было положено более века назад. Совершенно очевидны заслуги Саундерса в разработке теоретических основ теории теплопереноса при ламинарном течении, которые могут использоваться в расчетах при отличных от единицы числах Прандтля. Результаты этих расчетов хорошо согласуются с его экспериментальными данными, впервые выполненных им для этой жидкости, и с его несколько ранее проведенными измерениями для воздуха в широком диапазоне значений числа *Ra*. Таким образом, О. Саундерс был первым, кто точно определил начало перехода—того хорошо известного теперь степенного закона, определяющего турбулентный теплоперенос, а также эффект снижения вязкости при сильной турбулентности. Это те основы, за разработку которых следующие поколения исследователей остаются у него в долгу.